# INTERPOLATION OF MULTIPLE ZETA AND ZETA-STAR VALUES

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ABSTRACT. We define polynomials of one variable $t$ whose values at $t=0$ and 1 are the multiple zeta values and the multiple zeta-star values, respectively. We give an application to the two-one conjecture of Ohno-Zudilin, and also prove the cyclic sum formula for these polynomials.

## 1. INTRODUCTION

For integers $k_1 \geq 2$ and $k_2, \ldots, k_n \geq 1$, the multiple zeta value and the multiple zeta-star value (MZV and MZSV for short) are defined respectively as follows:

$$\zeta(k_1, \ldots, k_n) = \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}},$$

$$\zeta^\star(k_1, \ldots, k_n) = \sum_{m_1 \geq \cdots \geq m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

It is easy to see that an MZSV can be expressed as a linear combination of MZVs, and vice versa. Indeed,

$$\zeta^\star(k_1, \ldots, k_n) = \sum_{\mathbf{p}} \zeta(\mathbf{p}), \tag{1.1}$$

where $\mathbf{p}$ runs over all indices of the form

$$\mathbf{p} = (k_1 \square k_2 \square \cdots \square k_n)$$

in which each $\square$ is filled by the comma , or the plus +. If we denote by $\sigma(\mathbf{p})$ the number of + used in $\mathbf{p}$, we also have

$$\zeta(k_1, \ldots, k_n) = \sum_{\mathbf{p}} (-1)^{\sigma(\mathbf{p})} \zeta^\star(\mathbf{p}). \tag{1.2}$$

In this paper, we introduce the polynomial

$$\zeta^t(k_1, \ldots, k_n) = \sum_{\mathbf{p}} t^{\sigma(\mathbf{p})} \zeta(\mathbf{p}) \tag{1.3}$$

of one variable $t$. Note that $\zeta^0(\mathbf{k}) = \zeta(\mathbf{k})$ and $\zeta^1(\mathbf{k}) = \zeta^\star(\mathbf{k})$, i.e., this polynomial interpolates MZV and MZSV.

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There are many $\mathbb{Q}$-linear relations among MZVs, and some (families of) relations are named after their origin or their form. For example, the relation

$$\sum_{\substack{k_1\geq 2,k_2,\dots,k_n\geq 1,\\ k_1+\cdots+k_n=k}} \zeta(\mathbf{k}) = \zeta(k) \qquad (k > n \geq 1) \tag{1.4}$$

is called the sum formula, and the relation

$$\begin{aligned}&\zeta(a_1+1,\{1\}^{b_1-1},\dots,a_s+1,\{1\}^{b_s-1})\\ &\quad=\zeta(b_s+1,\{1\}^{a_s-1},\dots,b_1+1,\{1\}^{a_1-1}) \qquad (a_1,b_1,\dots,a_s,b_s\geq 1)\end{aligned}$$

is called the duality (here $\{\dots\}^l$ denotes the $l$ times repetition of the sequence in the curly brackets). It is also known that some of these relations have counterparts for MZSVs. For instance, the sum formula for MZSVs is

$$\sum_{\substack{k_1\geq 2,k_2,\dots,k_n\geq 1,\\ k_1+\cdots+k_n=k}} \zeta^{\star}(\mathbf{k}) = \binom{k-1}{n-1}\zeta(k) \qquad (k > n \geq 1), \tag{1.5}$$

while no complete counterpart of the duality has not been obtained (see [KO, L, Yz] and [TY] for some results on this problem).

We prove that some relations among MZVs and MZSVs can be extended to those among polynomials $\zeta^t$. An example is the sum formula for $\zeta^t$:

**Theorem 1.1.** *For any integers $k > n \geq 1$, we have*

$$\sum_{\substack{k_1\geq 2,k_2,\dots,k_n\geq 1,\\ k_1+\cdots+k_n=k}} \zeta^{t}(\mathbf{k}) = \left(\sum_{j=0}^{n-1}\binom{k-1}{j}t^j(1-t)^{n-1-j}\right)\zeta(k). \tag{1.6}$$

We study these polynomials from the viewpoint of harmonic algebra, the general setup of which was developed in [IKOO]. We briefly recall it in Section 2. In Section 3, we define an operator $S^t$ which gives an algebraic interpretation of (1.3) and a variant of the harmonic product compatible with $S^t$, and study some basic properties of them. Section 4 is devoted to the generalizations of the sum formula (Theorem 1.1 above) and the cyclic sum formula (Theorem 5.4). Finally, in Section 5, we consider the values at $t = \frac{1}{2}$ and discuss an application to the two-one conjecture of Ohno-Zudilin [OZ].

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## 2. General setup for harmonic algebra

Here we recall the formulation of harmonic algebra introduced in [IKOO].

Let $\mathfrak{A}$ be a commutative $\mathbb{Q}$-algebra, $\mathfrak{h}^1$ a non-commutative $\mathfrak{A}$-algebra of polynomials of a letter set $A$, and $\mathfrak{z}$ the $\mathfrak{A}$-submodule generated by $A$.

We assume that $\mathfrak{z}$ has a commutative $\mathfrak{A}$-algebra structure, not necessarily unitary, with product operation $\circ$ called the circle product. We define an action of $\mathfrak{z}$ on $\mathfrak{h}^1$ by

$$a \circ 1 = 0 \text{ and } a \circ (bw) = (a \circ b)w,$$

where $a, b \in A$ and $w \in \mathfrak{h}^1$ is a word (i.e. a monic monomial) and by $\mathfrak{A}$-linearity.

There are two $\mathfrak{A}$-bilinear products $*$ and $\star$ on $\mathfrak{h}^1$ defined by

$$\begin{gathered} w * 1 = 1 * w = w, \qquad w \star 1 = 1 \star w = w, \\ (aw) * (bw') = a(w * bw') + b(aw * w') + (a \circ b)(w * w'), \\ (aw) \star (bw') = a(w \star bw') + b(aw \star w') - (a \circ b)(w \star w'), \end{gathered}$$

for $a, b \in A$ and any words $w, w' \in \mathfrak{h}^1$. With each of these products, $\mathfrak{h}^1$ becomes a unitary commutative $\mathfrak{A}$-algebra, denoted by $\mathfrak{h}^1_*$ and $\mathfrak{h}^1_\star$ respectively.

**Example 2.1.** The basic example of the above setting is the case of $\mathfrak{A} = \mathbb{Q}$, $A = \{z_k \mid k = 1, 2, \ldots\}$, and $z_k \circ z_l = z_{k+l}$. Then the algebras $\mathfrak{h}^1_*$ and $\mathfrak{h}^1_\star$ give formal expressions of the harmonic product of MZVs and MZSVs respectively. Namely, set $\mathfrak{h}^0 = \mathfrak{A} \oplus \bigoplus_{k=2}^\infty z_k \mathfrak{h}^1$ and define $\mathbb{Q}$-linear maps $Z, Z^\star \colon \mathfrak{h}^0 \longrightarrow \mathbb{R}$ by

$$\begin{aligned} Z &\colon 1 \longmapsto 1, \quad z_{k_1} \cdots z_{k_n} \longmapsto \zeta(k_1, \ldots, k_n), \\ Z^\star &\colon 1 \longmapsto 1, \quad z_{k_1} \cdots z_{k_n} \longmapsto \zeta^\star(k_1, \ldots, k_n). \end{aligned}$$

Then $\mathfrak{h}^0$ is a $\mathbb{Q}$-subalgebra of $\mathfrak{h}^1$ with respect to both products $*$ and $\star$, and $Z \colon \mathfrak{h}^0_* \longrightarrow \mathbb{R}$ and $Z^\star \colon \mathfrak{h}^0_\star \longrightarrow \mathbb{R}$ are algebra homomorphisms.

Another interesting example is the $q$-analogue of the multiple zeta values introduced in [B]. See [IKOO, Example 2] for details.

## 3. The interpolating operator and $t$-harmonic product

### 3.1. The operator $S^t$.

**Definition 3.1.** Let $t$ be an indeterminate, and define an $\mathfrak{A}[t]$-linear operator $S^t$ on $\mathfrak{h}^1[t]$ by

$$S^t(1) = 1, \qquad S^t(aw) = aS^t(w) + t\, a \circ S^t(w) \tag{3.1}$$

(where $a \in A$ and $w \in \mathfrak{h}^1$ is a word).

Note that, if $\alpha$ is an element of a commutative $\mathfrak{A}$-algebra $\mathfrak{A}'$, one naturally obtain an $\mathfrak{A}'$-linear operator $S^\alpha$ on $\mathfrak{A}' \otimes_{\mathfrak{A}} \mathfrak{h}^1$ by substituting $\alpha$ into $t$.

**Example 3.2.** $S^0$ is the identity map on $\mathfrak{h}^1$, while $S = S^1$ is just the map considered in [IKOO], which satisfies $Z \circ S = Z^\star$ in the case of Example 2.1.

Here we show some basic properties of $S^t$. First, to describe $S^t(w)$ for a word $w$ explicitly, we introduce the following notation: For an integer $n \geq 0$, denote by $R_n$ the set of subsequences $r = (r_0, \ldots, r_s)$ of $(0, \ldots, n)$ such that $r_0 = 0$ and $r_s = n$. For such $r$ and a word $w = a_1 \cdots a_n \in \mathfrak{h}^1$, we define the word $\mathrm{Con}_r(w)$ (the contraction of $w$ with respect to $r$) by

$$\mathrm{Con}_r(w) = b_1 \cdots b_s, \qquad b_i = a_{r_i+1} \circ \cdots \circ a_{r_{i+1}}.$$

We also put $\sigma(r) = n - s$. Note that $\sigma(r)$ represents how many circle products there are in the definition of $\mathrm{Con}_r(w)$.

**Proposition 3.3.** *For any word $w \in \mathfrak{h}^1$ of length $n$, we have*

$$S^t(w) = \sum_{r \in R_n} t^{\sigma(r)} \mathrm{Con}_r(w).$$

*Proof.* It is clear that the above formula determines a map $S^t$ satisfying the relation (3.1). ☐

From Proposition 3.3, we see that $(S^t - 1)^n(a_1 \cdots a_n) = 0$ for $n \geq 1$. Therefore,

$$\log S^t = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (S^t - 1)^k \tag{3.2}$$

is well-defined as an operator on $\mathfrak{h}^1[t]$.

**Proposition 3.4.** *For a word $w$ of length $n$, we have*

$$(\log S^t)(w) = t \sum_{r \in R_n,\, \sigma(r)=1} \mathrm{Con}_r(w) = t(\log S)(w).$$

*Proof.* The following proof is based on an idea of Shingo Saito.

By Proposition 3.3, we can write

$$(S^t - 1)^k(w) = \sum_{r \in R_n} D(\sigma(r), k) t^{\sigma(r)} \mathrm{Con}_r(w),$$

where $D(\sigma(r), k)$ represents the number of all possibilities to obtain $\mathrm{Con}_r(w)$ from $w$ by taking non-trivial contractions $k$-times, which depends only on $\sigma(r)$ and $k$. In fact, $D(m, k)$ is equal to the cardinality of the set

$$\bigl\{(A_1, \ldots, A_k) \bigm| \{1, \ldots, m\} = A_1 \amalg \cdots \amalg A_k, A_1, \ldots, A_k \neq \emptyset\bigr\}.$$

By dividing this set into two subsets with respect to whether the element $m$ forms a singleton or not, we obtain the recurrence relation

$$D(m,k) = kD(m-1,k-1) + kD(m-1,k).$$

This relation immediately leads to the formula

$$\sum_{k=1}^{m} \frac{(-1)^{k-1}}{k} D(m,k) = \begin{cases} 1 & (m=1), \\ 0 & (m>1), \end{cases}$$

and then the first equality in the proposition follows. The second equality is deduced from the first. □

**Corollary 3.5.** (i) *We have $S^t = \exp(t \log S)$, where the right hand side is defined by the usual infinite series.*

(ii) *We have the equality $S^{t_1+t_2} = S^{t_1} \circ S^{t_2}$ as operators on $\mathfrak{h}^1[t_1,t_2]$. In particular, $S^{-t}$ is the inverse map of $S^t$.*

(iii) *For any word $w$ of length $n$, we have*

$$\frac{d}{dt} S^t(w) = (S^t \circ \log S)(w) = \sum_{r \in R_n,\, \sigma(r)=1} S^t(\mathrm{Con}_r(w)).$$

*Proof.* Clear from Proposition 3.4. □

3.2. **The $t$-harmonic product.** We define an $\mathfrak{A}[t]$-bilinear product $\overset{t}{*}$ on $\mathfrak{h}^1[t]$ by the recursive formula

$$1 * w = w * 1 = 1,$$

$$(aw) \overset{t}{*} (bw') = a(w \overset{t}{*} bw') + b(aw \overset{t}{*} w') + (1-2t)(a \circ b)(w \overset{t}{*} w')$$
$$+ (t^2 - t) a \circ b \circ (w \overset{t}{*} w')$$

for $a, b \in A$ and words $w, w' \in \mathfrak{h}^1$. We call it the $t$-harmonic product. Again, we may substitute any element $\alpha$ of a commutative $\mathfrak{A}$-algebra $\mathfrak{A}'$ to obtain an $\mathfrak{A}'$-bilinear product $\overset{\alpha}{*}$ on $\mathfrak{A}' \otimes_{\mathfrak{A}} \mathfrak{h}^1$. For example, $\overset{0}{*} = *$ and $\overset{1}{*} = \star$.

**Theorem 3.6.** *The product $\overset{t}{*}$ gives a commutative $\mathfrak{A}[t]$-algebra structure on $\mathfrak{h}^1[t]$ and the map $S^t$ is an algebra isomorphism of $(\mathfrak{h}^1[t], \overset{t}{*})$ onto $(\mathfrak{h}^1[t], *)$.*

*Proof.* Since $S^t$ is $\mathfrak{A}[t]$-linear and bijective, it is sufficient to show

$$S^t(w_1 \overset{t}{*} w_2) = S^t(w_1) * S^t(w_2) \qquad (w_1, w_2 \in \mathfrak{h}^1[t]). \tag{3.3}$$

This is proved in the same way as [IKOO, Theorem 1]. Namely, we prove (3.3) for words $w_1, w_2$ by induction on the length of $w_1 w_2$. For

$a, b \in A$ and words $w_1, w_2$, putting $W_1 = S^t(w_1)$ and $W_2 = S^t(w_2)$, one can show by direct calculation that

$$\begin{aligned} S^t(aw_1 \overset{t}{*} bw_2) =& a\big(W_1 * (bW_2)\big) + ta\big(W_1 * (b \circ W_2)\big) \\ &+ ta \circ \big(W_1 * (bW_2)\big) + t^2 a \circ \big(W_1 * (b \circ W_2)\big) \\ &+ b\big((aW_1) * W_2\big) + tb\big((a \circ W_1) * W_2\big) \\ &+ tb \circ \big((aW_1) * W_2\big) + t^2 b\big((a \circ W_1) * W_2\big) \\ &+ (1-2t)(a \circ b)(W_1 * W_2) - t^2 a \circ b \circ (W_1 * W_2). \end{aligned}$$

On the other hand, Lemma 1 and Lemma 2 of [IKOO] state that

$$\begin{aligned} &(a \circ W_1) * (bW_2) \\ &\quad = a \circ \big(W_1 * (bW_2)\big) + b\big((a \circ W_1) * W_2\big) - (a \circ b)(W_1 * W_2), \\ &(aW_1) * (b \circ W_2) \\ &\quad = b \circ \big((aW_1) * W_2\big) + a\big(W_1 * (b \circ W_2)\big) - (a \circ b)(W_1 * W_2), \\ &(a \circ W_1) * (b \circ W_2) \\ &\quad = a \circ \big(W_1 * (b \circ W_2)\big) + b \circ \big((a \circ W_1) * W_2\big) - (a \circ b) \circ (W_1 * W_2). \end{aligned}$$

Combining these formulas and the definition

$$(aW_1) * (bW_2) = a\big(W_1 * (bW_2)\big) + b\big((aW_1) * W_2\big) + (a \circ b)(W_1 * W_2),$$

we obtain

$$\begin{aligned} S^t(aw_1 \overset{t}{*} bw_2) &= (aW_1) * (bW_2) + t(a \circ W_1) * (bW_2) \\ &\quad + t(aW_1) * (b \circ W_2) + t^2(a \circ W_1) * (b \circ W_2) \\ &= (aW_1 + ta \circ W_1) * (bW_2 + tb \circ W_2) \\ &= S^t(aw_1) * S^t(bw_2) \end{aligned}$$

as desired. □

3.3. **Vanishing of an alternating sum.** As an application of the theory we have developed, we prove a natural generalization of the formula ([IKOO, Proposition 6])

$$\sum_{k=0}^{n} (-1)^k (a_1 \cdots a_k) * S(a_n \cdots a_{k+1}) = 0, \tag{3.4}$$

for $n \geq 1$ and $a_1, \ldots, a_n \in A$.

**Proposition 3.7.** *For $n \geq 1$ and $a_1, \ldots, a_n \in A$, we have*

$$\sum_{k=0}^{n} (-1)^k S^t(a_1 \cdots a_k) * S^{1-t}(a_n \cdots a_{k+1}) = 0. \tag{3.5}$$

*Proof.* When $n=1$, (3.5) amounts to the obvious equality $S^{1-t}(a)-S^t(a)=a-a=0$. Let $n\geq 2$ and assume the formula for $n-1$ holds. Since (3.5) holds at $t=0$ by virtue of (3.4), it suffices to show that the derivative of the left hand side with respect to $t$ vanishes. By the Leibniz rule, we have

$$\begin{aligned}
&\frac{d}{dt}\sum_{k=0}^{n}(-1)^k S^t(a_1\cdots a_k)*S^{1-t}(a_n\cdots a_{k+1})\\
&=\sum_{k=2}^{n}(-1)^k\sum_{i=1}^{k-1}S^t\big(a_1\cdots(a_i\circ a_{i+1})\cdots a_k\big)*S^{1-t}(a_n\cdots a_{k+1})\\
&\quad-\sum_{k=0}^{n-2}(-1)^k\sum_{i=k+1}^{n-1}S^t(a_1\cdots a_k)*S^{1-t}\big(a_n\cdots(a_{i+1}\circ a_i)\cdots a_{k+1}\big)\\
&=-\sum_{i=1}^{n-1}\left\{\sum_{k=0}^{i-1}(-1)^k S^t(a_1\cdots a_k)*S^{1-t}\big(a_n\cdots(a_{i+1}\circ a_i)\cdots a_{k+1}\big)\right.\\
&\qquad\left.+\sum_{k=i+1}^{n}(-1)^{k-1}S^t\big(a_1\cdots(a_i\circ a_{i+1})\cdots a_k\big)*S^{1-t}(a_n\cdots a_{k+1})\right\}.
\end{aligned}$$

Here the expression inside the bracket vanishes, for each $i=1,\dots,n-1$, by the induction hypothesis applied to the word $a_1\cdots(a_i+a_{i+1})\cdots a_n$. This completes the proof. $\square$

## 4. Application to the two-one formula

In this section, we apply our method to study a conjecture proposed in Ohno-Zudilin [OZ], called the two-one formula.

Let us be in the setting of Example 2.1. We denote the map $\mathfrak{h}^0[t]\to\mathbb{R}[t]$ obtained from $Z\colon\mathfrak{h}^0\to\mathbb{R}$ by tensoring $\mathbb{Q}[t]$ by the same letter $Z$. We put $Z^t=Z\circ S^t\colon\mathfrak{h}^0[t]\longrightarrow\mathbb{R}$ and $\zeta^t(k_1,\dots,k_n)=Z^t(z_{k_1}\cdots z_{k_n})$. Then $\mathfrak{h}^0[t]$ forms a subalgebra of $(\mathfrak{h}^1[t],\overset{t}{*})$, and $Z^t$ is an algebra homomorphism.

**Conjecture 4.1** (Two-one formula). *For integers $n\geq 1$ and $j_1\geq 1$, $j_2,\dots,j_n\geq 0$,*

$$\zeta^\star\big(\{2\}^{j_1},1,\{2\}^{j_2},1,\dots,\{2\}^{j_n},1\big)=2^n\zeta^{\frac12}(2j_1+1,2j_2+1,\dots,2j_n+1).$$

In [OZ], Ohno and Zudilin proved Conjecture 4.1 for $n=2$, and deduced the formula (for $j_1,j_2\geq 1$)

$$\zeta^\star\big(\{2\}^{j_1},1,\{2\}^{j_2},1\big)+\zeta^\star\big(\{2\}^{j_2},1,\{2\}^{j_1},1\big)=\zeta^\star\big(\{2\}^{j_1},1\big)\zeta^\star\big(\{2\}^{j_2},1\big)$$

as a corollary. In [TY], Tasaka and the author obtained a direct proof of this formula and its generalization:

**Theorem 4.2** ([TY, Theorem 1.2 (i)]). *For an integer $n \geq 1$ and non-negative integers $j_1, \ldots, j_n$ with $j_1, j_n \geq 1$, we have*

$$\sum_{k=0}^{n} (-1)^k \zeta^\star\big(\{2\}^{j_1}, 1, \ldots, \{2\}^{j_k}, 1\big) \zeta^\star\big(\{2\}^{j_n}, 1, \ldots, \{2\}^{j_{k+1}}, 1\big) = 0. \tag{4.1}$$

Therefore, assuming Conjecture 4.1, we should also have

$$\sum_{k=0}^{n} (-1)^k Z^{\frac{1}{2}}(z_{2j_1+1} \cdots z_{2j_k+1}) Z^{\frac{1}{2}}(z_{2j_n+1} \cdots z_{2j_{k+1}+1}) = 0. \tag{4.2}$$

In fact, this is an immediate consequence of Proposition 3.7. The consistency of the identities (4.1) and (4.2) give an evidence for the validity of Conjecture 4.1 for general $n$.

Now we investigate the algebraic structure of the right hand side of the two-one formula.

Put $y_j = 2z_{2j+1}$ for $j = 0, 1, 2, \ldots$, and let $\mathfrak{h}^{1,\mathrm{odd}}$ be the (non-commutative) subalgebra of $\mathfrak{h}^1$ generated by all $y_j$. We also put $\mathfrak{h}^{0,\mathrm{odd}} = \mathfrak{h}^{1,\mathrm{odd}} \cap \mathfrak{h}^0$.

**Proposition 4.3.** $\mathfrak{h}^{1,\mathrm{odd}}$ *forms a subalgebra of* $(\mathfrak{h}^1, \overset{1/2}{*})$. *Its multiplicative structure is determined recursively by*

$$y_i w \overset{1/2}{*} y_j w' = y_i(w \overset{1/2}{*} y_j w') + y_j(y_i w \overset{1/2}{*} w') - y_{i+j+1} \circ (w \overset{1/2}{*} w').$$

*Moreover, the $\mathbb{Q}$-linear map $\mathfrak{h}^{0,\mathrm{odd}} \longrightarrow \mathbb{R}$ defined by*

$$y_{j_1} \cdots y_{j_n} \longmapsto 2^n \zeta^{\frac{1}{2}}(2j_1+1, \ldots, 2j_n+1)$$

*is an algebra homomorphism.*

*Proof.* The first and the second assertions follows directly from the definition of the $\frac{1}{2}$-harmonic product. □

Proposition 4.3 and Conjecture 4.1 imply the following conjecture, which is interesting in its own right:

**Conjecture 4.4.** *The $\mathbb{Q}$-linear map $X\colon \mathfrak{h}^{0,\mathrm{odd}} \longrightarrow \mathbb{R}$ defined by*

$$X(y_{j_1} \cdots y_{j_n}) = \zeta^\star\big(\{2\}^{j_1}, 1, \{2\}^{j_2}, 1, \ldots, \{2\}^{j_n}, 1\big)$$

*is an algebra homomorphism with respect to the $\frac{1}{2}$-harmonic product.*

## 5. Sum formula and Cyclic sum formula

In this section, we continue to use the setting of Example 2.1 We want to prove the sum formula and the cyclic sum formula for the polynomials $\zeta^t(\mathbf{k})$. In addition, we also prove the equivalence of the families of linear relations obtained by substituting various numbers into $t$. For the latter purpose, we first make an elementary consideration on submodules of $\mathfrak{h}^1[t]$ which are closed under differentiation.

5.1. **Differential submodules of $\mathfrak{h}^1[t]$.** We call a $\mathbb{Q}[t]$-submodule $N$ of $\mathfrak{h}^1[t]$ a differential submodule if it is closed under the differentiation with respect to $t$, i.e., it satisfies

$$f(t) \in N \implies \frac{df}{dt}(t) \in N.$$

**Lemma 5.1.** *Let $N \subset \mathfrak{h}^1[t]$ be a differential submodule, $\mathfrak{A}$ a $\mathbb{Q}$-algebra and $\alpha \in \mathfrak{A}$. Then the $\mathfrak{A}[t]$-module $\mathfrak{A} \otimes_{\mathbb{Q}} N$ is generated by the $\mathfrak{A}$-submodule*

$$N_\alpha = \left\{ f(\alpha) \;\middle|\; f(t) \in \mathfrak{A} \otimes_{\mathbb{Q}} N \right\} \subset \mathfrak{A} \otimes_{\mathbb{Q}} \mathfrak{h}^1.$$

*Proof.* This is obvious from the expansion

$$f(t) = \sum_{k=0}^{n} a_k (t-\alpha)^k, \qquad a_k = \frac{1}{k!} f^{(k)}(\alpha) \in N_\alpha,$$

which holds for any $f(t) \in \mathfrak{A} \otimes_{\mathbb{Q}} N$. □

5.2. **Sum formula.** For integers $k > n \geq 1$, put

$$x_{k,n} = \sum_{\substack{k_1 \geq 2, k_2, \ldots, k_n \geq 1, \\ k_1 + \cdots + k_n = k}} z_{k_1} \cdots z_{k_n} \in \mathfrak{h}^1$$

and

$$P_{k,n}(t) = \sum_{j=0}^{n-1} \binom{k-1}{j} t^j (1-t)^{n-1-j} \in \mathbb{Q}[t].$$

We call the words appearing in $x_{k,n}$ the admissible words of weight $k$ and depth $n$.

**Lemma 5.2.** *For an integer $k \geq 2$, let $N_k^{SF} \subset \mathfrak{h}^1[t]$ be the $\mathbb{Q}[t]$-submodule generated by the elements $S^t(x_{k,n}) - P_{k,n}(t) z_k$ for all positive integers $n < k$. Then $N_k^{SF}$ is a differential submodule.*

*Proof.* One can deduce from Corollary 3.5 (iii) that

$$\frac{d}{dt} S^t(x_{k,n}) = (k-n) S^t(x_{k,n-1}),$$

since for each admissible word $w$ of weight $k$ and depth $n-1$, there are exactly $k-n$ admissible words of weight $k$ and depth $n$ such that $\mathrm{Con}_r(w') = w$ for some $r \in R_n$ (automatically satisfying $\sigma(r) = 1$). On

the other hand, we have

$$\begin{aligned}\frac{d}{dt}P_{n,k}(t) &= \sum_{j=1}^{n-1} j\binom{k-1}{j} t^{j-1}(1-t)^{n-1-j} \\ &\quad - \sum_{j=0}^{n-2}(n-1-j)\binom{k-1}{j} t^{j}(1-t)^{n-2-j} \\ &= \sum_{j=0}^{n-2}\left\{(j+1)\binom{k-1}{j+1} + (j+1-n)\binom{k-1}{j}\right\} t^{j}(1-t)^{n-2-j}.\end{aligned}$$

Since

$$\begin{aligned}(j+1)\binom{k-1}{j+1} + (j+1-n)\binom{k-1}{j} &= (j+1)\binom{k}{j+1} - n\binom{k-1}{j} \\ &= (k-n)\binom{k-1}{j},\end{aligned}$$

we obtain

$$\frac{d}{dt}P_{k,n}(t) = P_{k,n-1}(t).$$

Hence we have

$$\frac{d}{dt}\Big(S^t(x_{k,n}) - P_{k,n}(t)z_k\Big) = S^t(x_{k,n-1}) - P_{k,n-1}(t)z_k.$$

This completes the proof. □

*Proof of Theorem 1.1.* What we have to prove is that the differential submodule $N_k^{SF} \subset \mathfrak{h}^1[t]$ defined in Lemma 5.2 is contained in $\operatorname{Ker} Z$. By Lemma 5.1, it suffices to show that the $\mathbb{Q}$-submodule $(N_k^{SF})_0 = \{f(0) \mid f(t) \in N_k^{SF}\}$ of $\mathfrak{h}^1$ is contained in $\operatorname{Ker} Z$. This is, however, exactly what the sum formula (1.4) for usual MZVs says. Hence the proof is complete. □

**Remark 5.3.** One sees from the above proof that the sum formula (1.5) for MZSVs follows from the sum formula (1.4) for MZVs and vice versa. This fact was first proved (essentially) by Hoffman [H].

5.3. **Cyclic sum formula.** Let us recall the cyclic sum formulas for MZVs and MZSVs, established respectively by Hoffman-Ohno [HO]

and Ohno-Wakabayashi [OW]:

$$\sum_{l=1}^{n}\sum_{j=1}^{k_l-1}\zeta(k_l+1-j,k_{l+1},\dots,k_n,k_1,\dots,k_{l-1},j) = \sum_{l=1}^{n}\zeta(k_l+1,\dots,k_n,k_1,\dots,k_{l-1}), \tag{5.1}$$

$$\sum_{l=1}^{n}\sum_{j=1}^{k_l-1}\zeta^\star(k_l+1-j,k_{l+1},\dots,k_n,k_1,\dots,k_{l-1},j) = k\zeta(k+1). \tag{5.2}$$

Here $k_1,\dots,k_n\geq 1$ are not all 1, and $k=k_1+\cdots+k_n$.

The cyclic sum formula for the polynomials $\zeta^t$ is the following:

**Theorem 5.4.** *Let $n\geq 1$ and $k_1,\dots,k_n\geq 1$ be positive integers, and assume that $k_1,\dots,k_n$ are not all 1. Put $k=k_1+\cdots+k_n$. Then*

$$\begin{aligned}&\sum_{l=1}^{n}\sum_{j=1}^{k_l-1}\zeta^t(k_l+1-j,k_{l+1},\dots,k_n,k_1,\dots,k_{l-1},j)\\ &=(1-t)\sum_{l=1}^{n}\zeta^t(k_l+1,\dots,k_n,k_1,\dots,k_{l-1})+t^nk\zeta(k+1).\end{aligned} \tag{5.3}$$

We introduce some symbols: For any word $w=z_{k_1}\cdots z_{k_n}$ of length $n\geq 1$, we put

$$C(w)=\sum_{l=1}^{n} z_{k_l+1}z_{k_{l+1}}\cdots z_{k_n}z_{k_1}\cdots z_{k_{l-1}},$$
$$\Sigma(w)=\sum_{l=1}^{n}\sum_{j=1}^{k_l-1} z_{k_l+1-j}z_{k_{l+1}}\cdots z_{k_n}z_{k_1}\cdots z_{k_{l-1}}z_j.$$

Moreover, if $n\geq 2$, we set (putting $k_{n+1}:=k_1$)

$$\delta(w)=\sum_{l=1}^{n} z_{k_l+k_{l+1}}z_{k_{l+2}}\cdots z_{k_n}z_{k_1}\cdots z_{k_{l-1}}.$$

We extend them to $\mathbb{Q}[t]$-linear operators.

**Lemma 5.5.** *For any word $w$ of length $n\geq 1$, we have*

$$\frac{d}{dt}S^t\big(C(w)\big)=\begin{cases}0 & (n=1),\\ S^t\big(C\delta(w)\big) & (n\geq 2),\end{cases}$$
$$\frac{d}{dt}S^t\big(\Sigma(w)\big)=\begin{cases}(k-1)z_{k+1} & (n=1, w=z_k),\\ S^t\big(\Sigma\delta(w)\big)-S^t\big(C(w)\big) & (n\geq 2).\end{cases}$$

*Proof.* The case $n=1$ is immediate from the definition. Let $n\geq 2$ and write $w=z_{k_1}\cdots z_{k_n}$. Then, by Corollary 3.5 (iii), we have

$$\begin{aligned}\frac{d}{dt}S^t\big(C(w)\big)&=S^t\left(z_1\circ\sum_{l=1}^{n}\sum_{\substack{r=1\\ r\neq l-1}}^{n} z_{k_l}\cdots z_{k_r+k_{r+1}}\cdots z_{k_{l-1}}\right)\\&=S^t\left(z_1\circ\sum_{r=1}^{n}\sum_{\substack{l=1\\ l\neq r+1}}^{n} z_{k_l}\cdots z_{k_r+k_{r+1}}\cdots z_{k_{l-1}}\right)\\&=S^t\big(C\delta(w)\big).\end{aligned}$$

Similarly, we have $\frac{d}{dt}S^t\big(\Sigma(w)\big)=S^t(X)$, where

$$\begin{aligned}X&=z_1\circ\sum_{l=1}^{n}\sum_{j=1}^{k_l-1}\Bigg\{\sum_{\substack{r=1\\ r\neq l,l-1}}^{n} z_{k_l-j}\cdots z_{k_r+k_{r+1}}\cdots z_{k_{l-1}}z_j\\&\qquad\qquad + z_{k_l-j+k_{l+1}}\cdots z_{k_{l-1}}z_j+z_{k_l-j}\cdots z_{k_{l-1}+j}\Bigg\}\\&=z_1\circ\sum_{r=1}^{n}\Bigg\{\sum_{\substack{l=1\\ l\neq r,r+1}}^{n}\sum_{j=1}^{k_l-1} z_{k_l-j}\cdots z_{k_r+k_{r+1}}\cdots z_{k_{l-1}}z_j\\&\qquad\qquad+\sum_{\substack{j=1\\ j\neq k_r}}^{k_r+k_{r+1}-1} z_{k_r+k_{r+1}-j}z_{k_{r+2}}\cdots z_{k_{r-1}}z_j\Bigg\}\\&=\Sigma\delta(w)-C(w).\end{aligned}$$

This completes the proof of the lemma. $\square$

**Lemma 5.6.** *For an integer $k\geq 2$, let $N_k^{CSF}\subset\mathfrak{h}^1[t]$ be the $\mathbb{Q}[t]$-submodule generated by*

$$f_w(t)=S^t\big(\Sigma(w)\big)+(t-1)S^t\big(C(w)\big)-kt^n z_{k+1},$$

*where $w=z_{k_1}\cdots z_{k_n}$ runs over all words such that $n<k$ and $k=k_1+\cdots+k_n$. Then $N_k^{CSF}$ is a differential submodule.*

*Proof.* We extend $f_w(t)$ by linearity on $w$, e.g., $f_{w_1+w_2}(t)=f_{w_1}(t)+f_{w_2}(t)$. Then Lemma 5.5 implies that

$$\frac{d}{dt}f_w(t)=\begin{cases}0 & (n=1),\\ f_{\delta w}(t) & (n\geq 2).\end{cases}$$

Thus we obtain the lemma. $\square$

*Proof of Theorem 5.4.* Now the proof proceeds in the same way as the proof of Theorem 1.1. Namely, by Lemma 5.6 and Lemma 5.1, it is sufficient to show that $(N_k^{CSF})_0=\big\{f(0)\ \big|\ f(t)\in N_k^{CSF}\big\}$ of $\mathfrak{h}^1$ is

contained in $\operatorname{Ker} Z$, and this is the content of the cyclic sum formula (5.1) for usual MZVs. $\square$

**Remark 5.7.** The above proof implies that the formulas (1.4) and (1.5) are equivalent, as proved in [IKOO] and [TW].

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